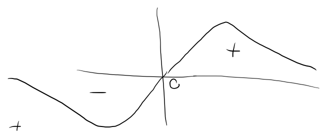
WT - Vorlesung 11

N(0,1):

- $f(x) > 0$ ✓
- $\int_{-\infty}^{\infty} f(x)\,dx = 1$ ohne Beweis

$$E(X) = \underbrace{\int_{-\infty}^{\infty} \underbrace{x}\cdot \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}}}_{\text{ungerade Funktion}} dx \quad \boxed{\int \text{konvergiert}} = 0$$



Formaler:

$$\int_{-\infty}^{\infty} \ldots = \int_{-\infty}^{0} x\cdot \frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{1}{2}x^2}\,dx + \int_{0}^{\infty} x\cdot\frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2}\,dx$$

$x = -y$
$dx = -dy$

$$= \int_{\infty}^{0} (-y)\frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}y^2}\cdot(-1)\,dy + \text{— '' —}$$

$$= \underbrace{-\int_{0}^{\infty} \frac{1}{\sqrt{2\pi}}\,\frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}y^2}\,dy + \text{— '' —}}_{=} = 0$$

$$\underline{Var(X)} = E(X^2) - (\underbrace{E(X)}_{0})^2 = E(X^2) = \int_{-\infty}^{\infty} x^2\cdot\frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^2 e^{-\frac{1}{2}x^2}\,dx$$

$$\boxed{\left(e^{-\frac{1}{2}x^2}\right)' = -\frac{1}{2}\cdot 2x\; e^{-\frac{1}{2}x^2} = -x\, e^{-\frac{1}{2}x^2}}$$

$$\boxed{\begin{array}{ll} v = -e^{-\frac{1}{2}x^2} & v' = x e^{-\frac{1}{2}x^2} \\ u = x & u' = 1 \end{array}}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{x}_{u}\cdot \underbrace{x e^{-\frac{1}{2}x^2}}_{v'}\,dx$$

part. int.

$$= \frac{1}{\sqrt{2\pi}}\left(\underbrace{\left[x\cdot\left(-e^{-\frac{1}{2}x^2}\right)\right]_{-\infty}^{\infty}}_{0} + \int_{-\infty}^{\infty} 1\cdot e^{-\frac{1}{2}x^2}\,dx \right) \quad \boxed{\int f\,dx = 1 \quad \text{Dichtefn.)}}$$

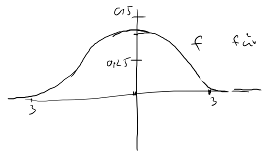
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\,dx = 1$$

Folie 6

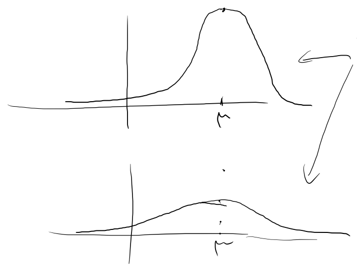
- $$\int_{-\infty}^{\overset{t_0}{\infty}} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) dx \underset{\boxed{y = \frac{x-\mu}{\sigma}}}{=} \int_{-\infty}^{\overset{\frac{t_0-\mu}{\sigma}}{\infty}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}y^2\right) dy = 1$$

$dy = \frac{1}{\sigma}dx$

↑ N(0,1)



f für N(0,1)



f für N(μ,σ)

$$E(X) = \int_{-\infty}^{\infty} x\cdot\frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) dx \underset{\boxed{y=\frac{x-\mu}{\sigma}}}{=} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \underbrace{(y\cdot\sigma+\mu)}_{x}\cdot\exp\left(-\frac{1}{2}y^2\right) dy$$

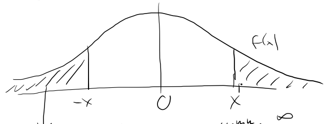
$X \sim N(\mu,\sigma^2)$

$dy = \frac{1}{\sigma}dx$

$$= \underbrace{\frac{1}{\sqrt{2\pi}}\cdot\sigma\int_{-\infty}^{\infty} \underbrace{y\cdot\exp\left(-\frac{1}{2}y^2\right)}_{\text{ungerade}}\,dy}_{0} + \mu\cdot\underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}y^2\right) dy}_{1} = \mu$$

$Var(X) = \ldots$ ähnlich $= \sigma^2$

$\underline{X \sim N(\mu,\sigma^2)}$ ... $\underline{P(X \le t_0)}$ = ... Folie 6 $= \underline{\Phi\left(\frac{t_0-\mu}{\sigma}\right)}$



$$\Phi(-x) = \int_{-\infty}^{-x} f(t)\,dt \overset{\text{symm.}}{=} \int_{x}^{\infty} f(t)\,dt = \underbrace{\int_{-\infty}^{\infty} f(t)\,dt}_{1} - \underbrace{\int_{-\infty}^{x} f(t)\,dt} = \underline{1 - \Phi(x)}$$

$$\Rightarrow \underline{\Phi(-x) + \Phi(x) = 1}$$

Bem: $P(X = t_0) = 0$ bei stetiger Zufalls var. X

$$\Rightarrow \underline{P(X \ge t_0)} = P(X > t_0) = \underline{1 - P(X \le t_0)}$$

Falls $\mu = 0$: $P(X \ge t_0) = P(X \le -t_0) =$

$\mu + (t_0 - \mu)$

$$\} \quad P(X \ge \mu + (t_0-\mu)) = P(X \le \mu - (t_0 - \mu)) = P(X \le 2\mu - t_0)$$

Bem 7.17: $P(\underline{\mu - k\cdot\sigma \le X \le \mu + k\sigma}) = P(X \le \mu + k\sigma) - P(X \le \mu - k\sigma)$

$$= \Phi\left(\frac{(\mu+k\sigma)-\mu}{\sigma}\right) - \Phi\left(\frac{(\mu-k\sigma)-\mu}{\sigma}\right) = \ldots$$

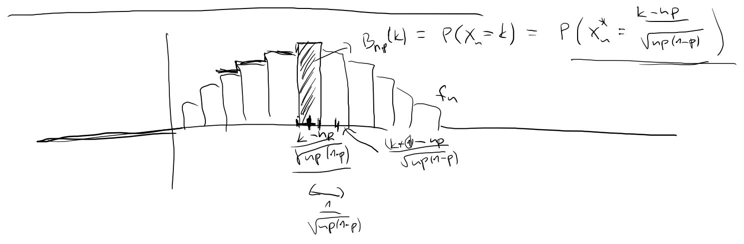
Bsp (gleichverteilt):

Würfeln mit „normalem Würfel"

$$X(x) = \begin{cases} -1, & x \text{ ungerade} \\ 1, & x \text{ gerade} \end{cases} \Rightarrow F_X(t) = P(X \le t) = \begin{cases} 0 & \boxed{t \le -1} \\ P(X \text{ ungerade}) = 1/2 & \boxed{-1 < t < 1} \\ 1 & \boxed{\text{falls } t \ge 1} \end{cases}$$

$$Y(x) = \begin{cases} -1, & \text{wenn } x \in \{1,2,3\} \\ 1, & \text{wenn } x \in \{4,5,6\} \end{cases} \rightsquigarrow F_Y(t) = F_X(t)$$

X und Y sind gleichverteilt



$$B_{np}(k) = P(X_n = k) = \underline{P\left(X_n^* = \frac{k-np}{\sqrt{np(1-p)}}\right)}$$

$$P(X \le x) = P\left(X^* \le \frac{x-\mu}{\sigma}\right) \overset{X\sim B_{n,p}}{=} P\left(X^* \le \frac{x-np}{\sqrt{np(1-p)}}\right) \underset{n \text{ groß}}{\overset{\text{Moivre-Laplace}}{\approx}} \Phi\left(\frac{x-np}{\sqrt{np(1-p)}}\right)$$